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The Equilibrium Range of Wind Wave Spectra: an Explanation Based on White Noise

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Abstract Laboratory experiments and field observations show that the equilibrium range of wind wave spectra presents a –4 power law when it is scaled properly. This feature has been attributed to energy balance in spectral space by many researchers. In this paper we point out that white noise on an oscillation system can also lead to a similar inverse power law in the corresponding displacement spectrum, implying that the –4 power law for the equilibrium range of wind wave spectra may probably only reflect the randomicity of the wind waves rather than any other dynamical processes in physical space. This explanation may shed light on the mechanism of other physical processes with spectra also showing an inverse power law, such as isotropic turbulence, internal waves, *etc*.

Key words wind wave spectrum; equilibrium range; –4 power law; white noise

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1 Introduction

Wind waves are always random, which is the result of the generating forces as well as the consequence of the dynamic processes in wave evolution that induce different kinds of instabilities (Wen and Yu, 1984; Sun and Ding, 1994). Therefore, one of the meaningful descriptions of wind waves is to employ the energy spectrum to represent the mean distribution of wave energy with respect to frequency or wave number by assuming that the wind wave is a stationary stochastic process. The part of wind wave spectrum, with wave numbers or frequencies between about three and five times the values at the peak in the spectrum, collapses onto a universal curve when the spectrum is properly scaled. This collapse was attributed to dynamic balance and so such part of spectra was referred to as the equilibrium range by Kitaigorodskii (1983) and Phillips (1985).

There has been considerable interest in the equilibrium range of wind wave spectra for the past several decades (Wen and Yu, 1984), because the corresponding micro-scale waves largely determine the momentum exchange between the atmosphere and the ocean (Makin *et al.*, 1995). It was found that short waves with wavelengths from centimeters to meters play a dominant role in supporting the form drag of waves, and the drag of developing seas depends crucially on the form of the wave spectrum in the corresponding high wave number or high frequency range (Makin *et al.*, 1995; Makin and Kudryavtsev, 1999; Kudryavtsev *et al.*, 1999). Moreover, this part of spectrum becomes of increasing practical significance in remote sensing (Wen and Yu, 1984; Yuan and Hua, 2005). For example, synthetic aperture radar (SAR) can be used to detect internal waves, submarine topography, *etc.* (Alpers, 1985; Gasparovic *et al.*, 1986; Liu *et al.*, 1985, 1998; Jin and Yuan, 1997). Yet, it is micro-scale waves that contribute most to the backscattering cross section in SAR (Valenzuela, 1978). Therefore, the spectral form of micro-scale waves and the corresponding mechanism involved are important to employing SAR imagery to detect internal waves, submarine topography, wind wave direction spectra, *etc.* (Jin and Yuan, 1997; Yuan and Hua, 2005).

Owing to its key role in remote sensing and the momentum exchange between the atmosphere and the ocean, many observations (Toba, 1973; Kawai *et al.*, 1977; Kahma, 1981; Forristall, 1981; Battjes *et al.*, 1987) and theoretical studies (Phillips, 1958; 1985; Kitaigorodskii, 1983) were conducted to explore the equilibrium range of wind wave spectra. However, so far the real causes of the universal behavior of the wind spectrum equilibrium range have not been revealed.

2 Review of Previous Work

The pioneering work in this field was carried out by

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Phillips (1958), who established the idea of saturation rather than equilibrium, that is to say, any excursion of the spectral density above a certain limit in the high frequency range is released immediately by breaking. Therefore, on purely dimensional grounds, the upper-limit spectral asymptote in the gravity-wave range is:

$$S(\omega) \propto g^2 \omega^{-5}, \quad (1)$$

where $S(\omega)$ is the frequency spectral density, ω the angular frequency and g the gravitational acceleration. In the following 25 years or so, the expression was utilized in some empirical spectra, such as the PM (Pierson and Moscowitz, 1964) and JONSWAP spectra (Hasselmann *et al.*, 1973).

However, later observational studies (Toba, 1973; Kawai *et al.*, 1977; Kahma, 1981; Forristall, 1981; Battjes *et al.*, 1987) confirmed the spectral form

$$\phi(\omega) \propto u_* g \omega^{-4} \quad (2)$$

for angular frequency ω in the gravity range above about three times of the spectral peak frequency, where u_* is the friction wind speed. Fig.1 shows a wind wave spectrum observed at the Physical Oceanography Laboratory's wind-wave-current flume in the Ocean University of China. The time series of surface elevation, used to estimate the spectrum in Fig.1, was sampled at a rate of 50 Hz and the sampling time was 5 min after the wind waves had reached a stable state. The wind speed, fetch and significant wave height corresponding to the spectrum in Fig.1 were 10 m s^{-1}, 40 m and 0.061 m, respectively. It is obvious that the spectral density in the equilibrium range is proportional to ω^{-4}. Subsequently researchers were motivated to propose other approaches to explaining the −4 power law shown in (2) since purely dimensional analysis cannot offer satisfactory explanations. It seems reasonable to attribute the cause of the spectral form shown in (2) to energy balance because the phenomenon is exhibited in spectral space, which was indeed followed by many authors (Wen and Yu, 1984; Kitaigorodskii, 1983; Phillips, 1985). Authoritative studies were conducted by Kitaigorodskii (1983) and Phillips (1985) who employed the spectral action transfer equation to explain the universal curve shown in (2). Kitaigorodskii (1983) proposed the existence of a Kolmogoroff-type equilibrium range in wind-generated waves in which the energy input from the wind is assumed to occur primarily at the energy-containing scales with dissipation restricted to much larger wave numbers. This then postulates the existence of a range of wave numbers over which the spectral flux divergence, wind input and dissipation rate are all negligible. Phillips (1985) assumed that the wave-wave interactions, energy input from the wind and the dissipation by wave-breaking are all important in the equilibrium range. The assumptions of Phillips and Kitaigorodskii are obviously in disagreement; however, both of them can be used to derive the spectral form similar to that of Eq. (2). It is difficult to prove which one of the two is more justifiable. Moreover, Banner (1990) pointed out that different spectral forms of the equilibrium range could be obtained when two different terms of the three (the wave-wave interaction, energy input and dissipation) were chosen to be in balance. Belcher and Vassillcos (1997) tried another way to explain why the frequency spectrum of wind-waves over the equilibrium range is proportional to ω^{-4}. They assumed that breaking waves were self-similar and scale-invariant and then found that $S(\omega) \propto \omega^{-4}$ in the equilibrium range. The dynamical balance in the spectral space employed by Phillips was used by Belcher and Vassillcos (1997) too.

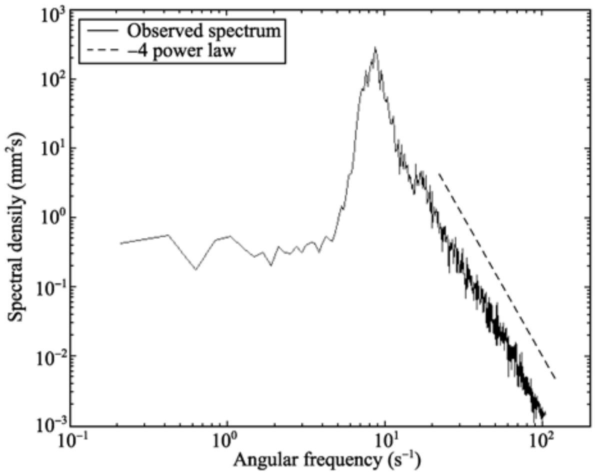

Fig.1 A typical wind wave spectrum observed at Physical Oceanography Laboratory's wind-wave-current flume in the Ocean University of China. The wind speed, fetch and significant wave height are 10 m s^{-1}, 40 m and 0.061 m, respectively.

So far none of all the existing explanations concerning the energy balance in spectral space have considered the problem directly on the basis of wave dynamics. Here we try to interpret the inverse power law in the equilibrium range of wind wave spectra through wave dynamics in physical space. The main idea of the present paper is presented in section 3; some remarks are given in section 4.

3 Relationship Between White Noise and the −4 Power Law in the Equilibrium Range

It is well known that wind waves are random, which is the result of the generating forces as well as the consequences of dynamic processes in wave evolution that induce different kinds of instabilities (Wen and Yu, 1984). But on the other hand, wind waves are not completely in random form, the general behavior of wind waves appears to be definite. So it seems more reasonable to divide the actual wind waves based on two scales: the dominant waves and the short gravity waves. The dominant waves denote those waves which have relatively steady structure and correspond to the energy-containing range of the wind wave spectrum. The short waves are those that fall into the equilibrium range and are modulated by the dominant waves (Wen and Yu, 1984). Compared with the dominant waves, the short gravity waves show more randomicity due to the direct influence from wind and wave breaking. Therefore it can be speculated that the inverse power law in the equilibrium range should be caused just by randomicity of the short gravity waves. If the speculation is true, the key question is, subsequently, how to characterize the randomicity.

White noise is a common stochastic process that can be presented as:

$$X(t)=\sum_{i=1}^{\infty} a\cos(\omega_i t+\varepsilon_i)\,, \tag{3}$$

where the amplitude, a is a constant, ω_i is the angular frequency, ε_i is the initial random phase distributed uniformly in the interval [0, 2π]. White noise is frequently utilized to characterize many processes that are full of randomicity, therefore the randomicity of short gravity waves, accordingly the inverse power law in the equilibrium range, may also be related to a white noise process.

First, by ignoring the influence from both the orbital motion of the dominant waves and the ambient currents, neglecting the corresponding nonlinear terms and assuming that the forces on the short gravity waves are band-limited white noise, the most simplified governing equation of the surface fluctuation due to short gravity waves can be expressed as:

$$\frac{\partial^2 \zeta}{\partial t^2}=\sum_{i=1}^{\infty} a\cos(\omega_i t+\varepsilon_i)\,, \tag{4}$$

where ζ represents the surface displacement of the short gravity waves at a fixed point and the angular frequency ω_i is limited over a certain band. Considering that both $\frac{\partial \zeta}{\partial t}$ and ζ are stochastic processes with zero mean values, we can obtain ζ through integrating (4) twice:

$$\zeta=-\sum_{i=1}^{\infty} \frac{a}{\omega_i^2}\cos(\omega_i t+\varepsilon_i)\,. \tag{5}$$

It is obvious that the power spectra of ζ is proportional to ω^{-4}, which means that if the forces on the simplified surface fluctuations are pure white noise processes, the corresponding frequency spectra present the inverse power law as shown in (2).

Practically the short gravity waves are always superimposed on the dominant waves, it is more reasonable to consider the influence of the white noise on the dominant waves. The dominant waves actually include many waves with different wave numbers or frequencies. Here for simplicity, a regular sinusoidal wave is chosen to represent the dominant waves. Ignoring the corresponding nonlinear terms and the Doppler shifting from ambient currents, the influence of white noise on a sinusoidal wave can be characterized as:

$$\frac{\partial^2 \zeta}{\partial t^2}+\omega_0^2\zeta=\sum_{\substack{i=1\\ \omega_i>\omega_0}}^{\infty} a\cos(\omega_i t+\varepsilon_i)\,, \tag{6}$$

where ω_0 is the angular frequency of the sinusoidal wave. In (6) $\omega_i>\omega_0$ is required because only the higher frequency range is of our interenst. From Eq. (6) we can obtain:

$$\zeta=b\cos(\omega_0 t+\varepsilon_0)+\sum_{\substack{i=1\\ \omega_i>\omega_0}}^{\infty} \frac{a}{\omega_0^2-\omega_i^2}\cos(\omega_i t+\varepsilon_i)\,, \tag{7}$$

where the first term on the right hand side is the general solution of the corresponding homogeneous equation and the second term, the particular solution of the non-homogeneous equation. Obviously, the frequency spectrum corresponding to the particular solution is:

$$S(\omega)\propto\frac{1}{\left(\omega_0^2-\omega_i^2\right)^2}\,, \tag{8}$$

which is shown in Fig.2, where the peak frequency of the observed spectrum is chosen as ω_0, *i.e.*, ω_0=8.6 s^{-1}. From the figure, we can see that the spectrum (8) agrees satisfactorily with the observed spectrum in the equilibrium range. Moreover, in the range of the observed spectrum from the peak frequency to twice of the peak frequency, (8) also offers similar variations.

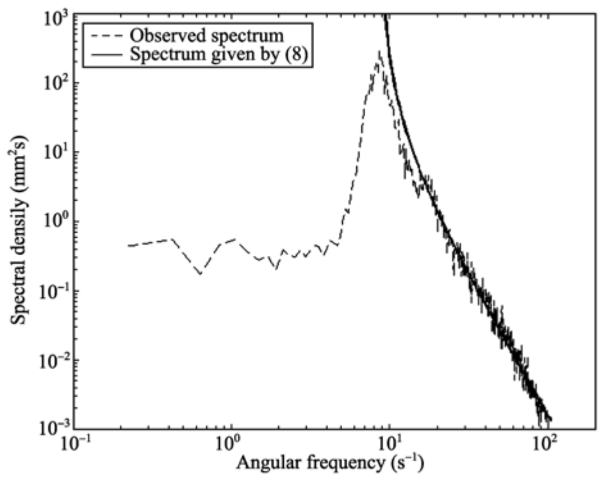

Fig.2 Spectrum expressed in (8) and the observed spectrum. ω_0 in (8) is chosen as the peak frequency of the observed spectrum, *i.e.*, ω_0=8.6 s^{-1}. The parameters of the observed spectrum are the same as those in Fig.1.

Generally speaking, different spectral behaviors can be obtained if white noise acts on different dynamical systems. The two cases mentioned above successfully achieve the primary behaviors of the wind wave spectra in the equilibrium range although they take only simple and linear dynamical systems into consideration. It implies that the inverse power law in the equilibrium range of wind wave spectra may probably only reflect the randomicity of wind waves rather than any other dynamical processes in physical space.

4 Remarks

Laboratory experiments and field observations show that the equilibrium range of wind wave spectra presents a −4 power law when it is scaled properly (Toba, 1973; Kawai *et al.*, 1977; Kahma, 1981; Forristall, 1981; Battjes *et al.*, 1987), which is attributed to energy balance in spectral space (Kitaigorodskii, 1983; Phillips, 1985). It is found in this study that the effect of white noise on a linear dynamical system can also lead to a similar inverse power law for the spectrum of the corresponding displacement. Compared with former explanations, the effect of white noise, leading to the inverse power law, is introduced in a simple and clear way. It also suggests that the inverse power law probably only reflects the randomicity of short gravity wind waves rather than any other dynamical processes in physical space. An assumption is introduced in this paper, *i.e.* the randomicity of short wind waves can be characterized by white noise. So the conclusions of this paper are only tentative rather than final.

As is well known, power spectra that exhibit inverse power law behavior are ubiquitous in dynamical phenomena in nature (Selvam *et al.*, 2000), such as atmospheric flow, stock market price fluctuations, population growth, internal waves, *etc*. These phenomena are dominated by distinct dynamical systems, but the corresponding power spectra show similar forms. Although some authors have attributed the causes to chaos or fraction, we hope this paper may offer some inspirations for studying those processes.

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